Rational Number Representation by the Approximate Number System

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The approximate number system (ANS) enables organisms to represent the approximate number of items in an observed collection, quickly and independently of natural language. Recently, it has been proposed that the ANS goes beyond representing natural numbers by extracting and representing rational numbers (Clarke & Beck 2021a). Prior work has demonstrated that adults and children discriminate ratios in an approximate and ratio-dependent manner, consistent with the hallmarks of the ANS. Here, we use a well-known "connectedness illusion" to provide evidence that these ratio-dependent ratio discriminations are (a) based on the perceived number of items in seen displays (and not just non-numerical confounds), (b) are not dependent on verbal working memory, or explicit counting routines, and (c) may involve representations with a part-whole (or subset-superset) format, like a fraction, rather than a part-part format, like a ratio. These results support and refine the hypothesis that the ANS represents full-blown rational numbers.

1. Introduction

Humans, and many non-human animals, possess a *number sense* (Dehaene 1997) or *approximate number system* (ANS), that represents number. Evidence for this system is found in myriad creatures – including fish (e.g., Agrillo et al. 2008), rats (e.g., Platt & Johnson 1971), pigeons (e.g., Emmerton et al., 1997), monkeys (e.g., Brannon and Terrace, 1998), human infants (e.g., Xu & Spelke 2000), pre-numerate human children (e.g., Mix et al., 2002), human adults whose language lacks precise number words (Pica et al. 2004), as well as human adults with a formal math education (e.g., Barth et al. 2003). In addition, the system has been found to support a diverse range of numerical computations – for instance, ordinal comparisons (Temple and Posner, 1998), the ability to identify two collections as equinumerous (Barth et al. 2003), number estimations (Cordes et al. 2001), as well as addition (McCrink & Wynn 2004), subtraction (Barth et al. 2006), multiplication (Qu et al. 2021; McCrink & Spelke, 2010) and division operations (McCrink & Spelke, 2016; Szkudlarek et al., 2022). Consequently, an orthodox view in cognitive science is that the ANS is widespread in nature, operates independently of natural language, and enables humans and other organisms to process numbers throughout the lifespan, albeit approximately and in accord with Weber's Law.

Despite this prevailing orthodoxy, relatively little attention has been paid to the kinds of number that the ANS represents. The vast majority of ANS research focusses exclusively on the system's representation and discrimination of natural numbers; for instance, the system's capacity to discriminate *8* items from *16* items, or to add the number of whole items in two collections together. While some have proposed that the ANS goes beyond representing natural numbers by representing real numbers quite generally (e.g., Gallistel & Gelman 2000; Dramkin & Odic 2021) arguments in favour of this conjecture have been roundly criticized (Laurence & Margolis 2005). Consequently, it is tempting to suppose that the ANS is exclusively in the business of representing whole or natural numbers.

A recent article by Clarke and Beck (2021a) questions this suggestion. The authors of this paper agree that there is little reason to think that the ANS represents irrational numbers, and hence the real numbers in general. However, they propose that we have provisional reason to think the ANS goes beyond representing natural numbers (like 1, 2, 3, 4, 5…), by representing rational numbers, which can be expressed as a ratio or fraction among natural numbers (e.g., 1:3 or ¼). To this end, they observe that non-human animals, human infants, and young children are sensitive to proportional relations. For example, six-month-old infants, who have been habituated to a specific ratio of blue to yellow elements, look significantly longer at arrays with a novel ratio (McCrink and Wynn, 2006). Slightly older infants rely on the ratio of preferred to non-preferred lollipops in two collections to decide which of two blind samples is more likely to contain a desired lollipop (Denison & Xu 2014; see also: Xu & Garcia 2008; Xu & Denison 2009; Denison & Xu 2010; Fontanari et al. 2014; and Kayhan et al. 2018). Similarly, non-human apes (Rakoczy et al. 2014; Eckert et al. 2018) and monkeys (Drucker et al. 2016; Tecwyn et al. 2017) can reliably distinguish arrays based on the ratio of preferred to non-preferred food items.

Sensitivity to ratios or probabilities so early in development is surprising given that learning symbolic notations for rational numbers (e.g., fractions, ratios, and percentages) is notoriously difficult for children (e.g., Lortie-Forgues, Tian, & Siegler 2015). In fact, Piaget (Inhelder & Piaget 1958; Piaget & Inhelder 1951/1975) argued that proportional reasoning is only possible once children understand formal operations. However, detailed examination of children's proportional reasoning implicates an intuitive understanding of non-symbolic proportional relations that emerges long before the capacity to grasp formal operations obtains (e.g., Siegler et al. 2013; Matthews & Chesney 2015; Lewis et al. 2016; Matthews et al. 2016; Bhatia et al. 2020; Binzak & Hubbard 2020; O'Grady & Xu, 2020). For example, Szkudlarek and Brannon (2021) presented 6–8-year-old children with pairs of gumball machines, each containing blue and white 'gumballs' (dots) or blue and white Arabic numerals specifying the number of blue and white gumballs that machine contained. The children were tasked with reporting which gumball machine was most likely to randomly produce a single gumball of a preferred color. They were significantly above chance in both symbolic and non-symbolic versions of the task (i.e., whether the numbers of gumballs were represented using Arabic numerals or depicted as a collection of dots), even though they had not begun to study fractions in school and still could not make or understand precise fraction comparisons. Since subsequent analyses revealed that they could not have relied exclusively on simpler heuristics (e.g., choosing machines which contained the larger absolute number of desired gumballs, or the smaller absolute number of undesired gumballs), the results of this study indicate that children were discriminating the ratio of blue to white gumballs in each machine and subsequently comparing those ratios, with their accuracy predicted by Weber's Law (i.e., the ratio between the ratios of gumballs in each machine).

In this latter respect, children's ratio discriminations mirrored the performance profile of the numerical discriminations, facilitated by the ANS. But is ratio-dependent performance in a ratio-comparison task, such as that used by Szkudkarek and Brannon, evidence that the ANS represents rational numbers as Clarke and Beck (2021a) suggest? An alternative possibility would be that participants were relying on continuous properties of the display when comparing ratios. For instance, in Szkudlarek and Brannon's (2021) study, the size and spacing of the dots was not controlled for. Thus, it is possible that children's discriminations were based on an appreciation for the ratios/proportions among continuous (non-numerical) properties of the stimuli: for instance, between the total surface area of blue dots on the screen and the total surface area of white dots on the screen. Indeed, similar worries could be raised in relation to many of the abovementioned ratio experiments and may seem pressing given existing critiques of the ANS which assert that its comparisons simply pertain to non-numerical values (Gebuis et al. 2016; Leibovich et al. 2017; though see Park et al. 2021). While it might be replied that the children from Szkudlarek and Brannon's (2021) study performed equally well in a symbolic version of the task, which effectively eliminates these confounds, symbolic number comparisons may rely on orthogonal processes (Gomez 2021; Krajsci et al. 2022). Thus, Experiment 1 of the present treatment sought to adjudicate these concerns directly. Using a 'connectedness illusion' (Franconerri et al. 2009; He et al. 2009), which systematically alters perceived number without significantly affecting low-level confounds, we provide evidence that non-symbolic ratio comparisons are based on natural number, and not just non-numerical properties of seen collections (e.g., their total surface area, density, or average brightness).

A second argument that ratio comparisons (of the sort investigated by Szkudlarek and Brannon [2021]) need not imply that the ANS represents rational numbers is that performance could reflect domain general analogical reasoning. For example, Hecht et al. (2021) argue that "The ANS represents natural numbers, and there are separate, non-numeric processes that can be used to represent ratios across a wide range of domains, including natural numbers." Such claims admit various interpretations. But a strong reading, which would be particularly problematic for Clarke and Beck's hypothesis, will maintain that while natural numbers are extracted by the ANS, ratios or fractions among these natural numbers are simply derived by domain general resources involved in flexible thought. For instance, having represented that there are 8 or 8ish blue items in one subset or collection and 10 or 10ish red items in another subset or collection, participants might appreciate that these numbers stand in an 8 to 10 ratio to one another, by employing post-perceptual cognitive resources involved in domain general reasoning. Indeed, Dramkin and Odic (2021) note that, if this were so, the resulting ratio discriminations (which, as we have seen, conform to Weber's Law) might inherit their characteristic imprecision from the imprecision of the ANS's representations of natural number which serve as input to this central cognitive process of ratio discrimination. To adjudicate this concern, we conducted a second experiment in which participants engaged in a secondary verbal shadowing task, while performing the ratio comparison task tested in Experiment 1. While this secondary task taxes verbal working memory, in ways that are known to hinder verbal arithmetic (Hecht 2002; Seitz & Schumann-Hengsteler 2002; Imbo & Vandierendonck 2007) and analogical reasoning (Waltz et al. 2000), we find that participants' performance in this second experiment is largely unaffected as compared to those tested in Experiment 1.

Finally, some researchers object that even if extant evidence supports the view that the ANS represents *ratios*, ratios are not rational numbers. Critiques of this type take different forms (see: Gomez 2021; Lyons 2021). For instance, Ball (2021) objects that, unlike genuine rational numbers (e.g., numbers expressed as fractions or decimals), ratios lack an additive structure. So, while fractions and decimals are apt to be added and subtracted (e.g., ¼ + ¼ = ½), ratios are not like this: if the blue and red balls in one box stand in a 1:3 ratio, while the blue and red balls in a second box stand in a 2:3 ratio, we have no way of knowing what ratio the blue and red balls would stand in if they were both poured into a single box without first establishing the absolute number of blue and red balls in either box. Somewhat distinctly, Peacocke (2021) objects that genuine rational number representation by the ANS requires that values be placed on a single "rational number line". Clarke and Beck (2021b) have responded to these concerns on broadly philosophical grounds. But while the current study was not designed to address these concerns directly, we conjecture that there are *empirically tractable* differences between representations of ratios and fractions. Perhaps most notably: Where ratio representations specify a part-part relation (i.e., to say that there is a 1:3 ratio of blue to red balls in a bucket is to say that for every ball in the blue subset of the collection there are three balls in the red subset of the collection), proportions or fractions encode a part-whole relation (i.e., to say that ¼ of the balls are blue is to say that for every ball in the blue subset of the collection there are four *blue or non-blue* balls in the superset). We present posthoc analyses, showing that participants' pattern of performance in Experiments 1 and 2 is straightforwardly predicted by their employment of representations with part-whole structure, which mirrors the form of a fraction or proportion rather than a literal ratio. While alternative explanations are considered, our discussion serves as a proof of principle, showing how progress can be made investigating the format of sub-linguistic proportional representations.

Experiment 1

Previous research shows that humans can discriminate non-symbolic ratios, quickly, efficiently, and without recourse to explicit counting, albeit imprecisely and in accord with Weber's Law. However, it remains unclear whether these ratio comparisons pertain to ratios among the *number* of perceived elements in a display, or continuous properties of the collections compared. For instance, if all elements in a collection are of a single size, discriminating the ratios among relevant elements could be based on an appreciation for their non-numerical properties, like their total surface areas (but see Park et al. 2021), or perhaps their density, brightness, and convex hull. Indeed, this concern is pressing, given prior work suggesting that children find continuous proportions easier to compare than proportions based on discrete items (e.g., Boyer, Levine, & Huttenlocher, 2008; Jeong et al., 2007; Singer-Freeman & Goswami, 2001; Spinillo & Bryant, 1999; Park, Viegut & Matthews, 2020).

To adjudicate the above concern, our first experiment takes advantage of the 'connectedness illusion' (Franconerri et al. 2009). In the connectedness illusion, items (e.g., circles) are connected with thin lines, effectively turning pairs of items into single dumbbell-shaped objects. Under these conditions, observers enjoy a significant reduction in the perceived number of items in the display, as compared with an unconnected condition, in which the same lines are present but do not connect circles (He et al. 2009) or contain small breaks (Franconerri et al. 2009; see also Kirjakovski & Matsumoto 2016). Crucially, these results persist, even when participants are instructed to ignore the lines and to focus only on the circles in the displays. This is striking, since arrays containing thin lines which *connect* circles barely differ with respect to their total surface area, density, convex hull, or average brightness, when compared with arrays containing an identical number of circles and free-floating lines. Thus, the reduction in perceived number appears to reflect the fact that the ANS functions to track and enumerate discrete and bounded whole objects, rather than their continuous properties (Clarke & Beck 2021a).

We reasoned that if approximate ratio comparisons are based on prior representations of whole numbers (i.e., the natural number of items in relevant subsets and/or supersets), produced by the ANS, these comparisons should be systematically affected by connectedness. Thus, we replicated Szkudlarek and Brannon's (2021) gumball task (described previously) but with an added manipulation: We connected pairs of gumballs to systematically manipulate their perceived number, and to thereby influence the favorability of the ratios in which they featured, as compared to a baseline condition in which gumballs were left unconnected.

To illustrate, imagine that you are presented with an array containing 16 blue items and 10 red items wherein blue items are the favorable (target) color (i.e., instructions require choosing the machine with the best odds of randomly returning a single blue gumball in a blind draw). In a baseline condition, with no connections, 16:10 would be a highly favorable ratio since 16 is almost double 10. But if the 16 blue items were connected into 8 pairs this would reduce the perceived number of blue items and consequently reduce the favorability of the ratio. By contrast, if red items are connected this would increase the perceived favorability of the ratio because it would lead to a reduction in the perceived number of unfavorable (non-target) items and, thus, lead to an increase in the perceived ratio of blue to red items. In either case, if participants are simply relying on the ratio among continuous properties, connectedness should have a negligible effect because this manipulation does not significantly change the total surface area, perimeter, convex hull, or average brightness of red and/or blue items.

We predicted that, replicating Szkudlarek and Brannon (2021), the ratio among the ratios in the two arrays would modulate performance, as measured in a baseline condition where neither stimulus contained connected items and lines were free-floating. In addition, we hypothesized that, if ratio comparisons were based on the prior extraction of the natural number of blue and red items in the display (by the ANS), performance would be systematically influenced by the connectedness of the items (even when subjects tried to ignore these connections and focus only on the dots). Thus, when favorable items in a ratio were connected, this would reduce the perceived number of favorable items, and in turn, reduce the probability with which that ratio was chosen compared to a baseline condition with no connections. Meanwhile, connections among unfavorable items in a ratio, should reduce the perceived number of those items and, thus, increase the perceived favorability of the ratio compared to a baseline condition. We, thus, predicted that perceived number would be influenced by the number of bounded (roughly 'Spelke') objects in seen collections (Spelke 1990; c.f. Green 2018), and that connectedness would systematically influence the perceived ratio of the favorable to unfavorable items in the array.

Method

In this and subsequent experiments, the sample size, primary dependent variables, and key statistical tests were determined prior to data collection and were pre-registered. Those pre-registrations can be accessed here: https://osf.io/6gbcz/.

Participants

39 adults participated in this experiment (Mean age = 22.57 years, SD = 5.76; 24 males, 15 females). Participants were recruited online, using Prolific, tested online, and paid a modest sum for their participation. Participants provided informed consent to a protocol approved by the Institutional Review Board in a large research university. Each participant was required to confirm that they had normal or corrected-to-normal vision, were completing the task on a computer with a screen-size of 13-16 inches diagonal length and did not suffer from color blindness (color sightedness was also independently verified in an online test). Additional preregistered exclusion criteria were 1) a side bias with greater than 65% of responses to the left or right side, 2) failure to respond to more than 10% of trials, 3) at or below chance performance on baseline trials, 4) mean accuracy less than 3 times the interquartile range of the sample. These criteria were chosen to ensure that participants were engaging with the task, and not answering randomly, or employing some task-irrelevant strategy. Following the pre-

registered criteria, 3 participants were excluded because they did not complete all the experimental blocks; 10 participants were removed because they did not perform above chance expectations on baseline trials where all elements were unconnected. The final sample size was 26 (Mean age = 22.47, SD = 1.89; 18 males, 8 females).

Stimuli

Stimuli were composed of pseudo-randomly arranged dots with thin lines either connecting two adjacent dots or free floating within white squared panels against a neutral grey background. A total of 105 different dot arrays were created offline. In each stimulus array, there were both blue dots (RGB: 0 0 255) and red dots (RGB: 255 0 0). Within a single array, the absolute number of blue and red dots varied near-uniformly from 5 - 36 and the ratio of blue to red dots varied from 0.667 – 1.5. On all trials the red and blue dots were spatially intermixed, as illustrated in Figure 1.

Lines were generated with random orientations and spatial positions and matched the colors of the dots that they connected. Stimuli were constructed such that they contained the minimum number of blue lines that would be needed to ensure that each blue dot was connected with one other. However, if a stimulus contained an uneven number of to-be-connected dots, a triplet was created. Thus, a stimulus with 8 blue dots, would warrant 4 thin blue lines to connect the 8 dots into 4 dumbbell-shaped objects and a stimulus with 9 red dots would warrant 5 red lines to connect the 9 dots into 3 pairs and one triplet. Connecting lines were not permitted to cross each other.

There were two kinds of stimuli in this experiment. In c*onnected* stimuli the blue or red dots were connected by blue lines into pairs (and a single triplet if there were an odd number of dots). *Unconnected* stimuli contained the identical number of lines however the lines were free-floating and thus did not connect any of the dots. This enabled us to manipulate the perceived number of blue or red items, independently of their continuous confounds (e.g., the total red/blue surface area presented on the screen), thereby altering the ratio among numbers of blue to red bounded items without altering the ratios among their continuous confounds.

As shown in Fig 1, there were 5 different trial types: 1) trials in which both red and blue items were left unconnected in both arrays (the baseline condition); 2) trials in which favorable items in the less-favorable ratio were connected; 3) trials in which un-favorable items in the more-favorable ratio were connected; 4) trials in which un-favorable items in the less-favorable ratio were connected; 5) trials in which favorable items in the more-favorable ratio were connected. Each participant completed 280 trials in total, with 56 trials at each of the five trial types.

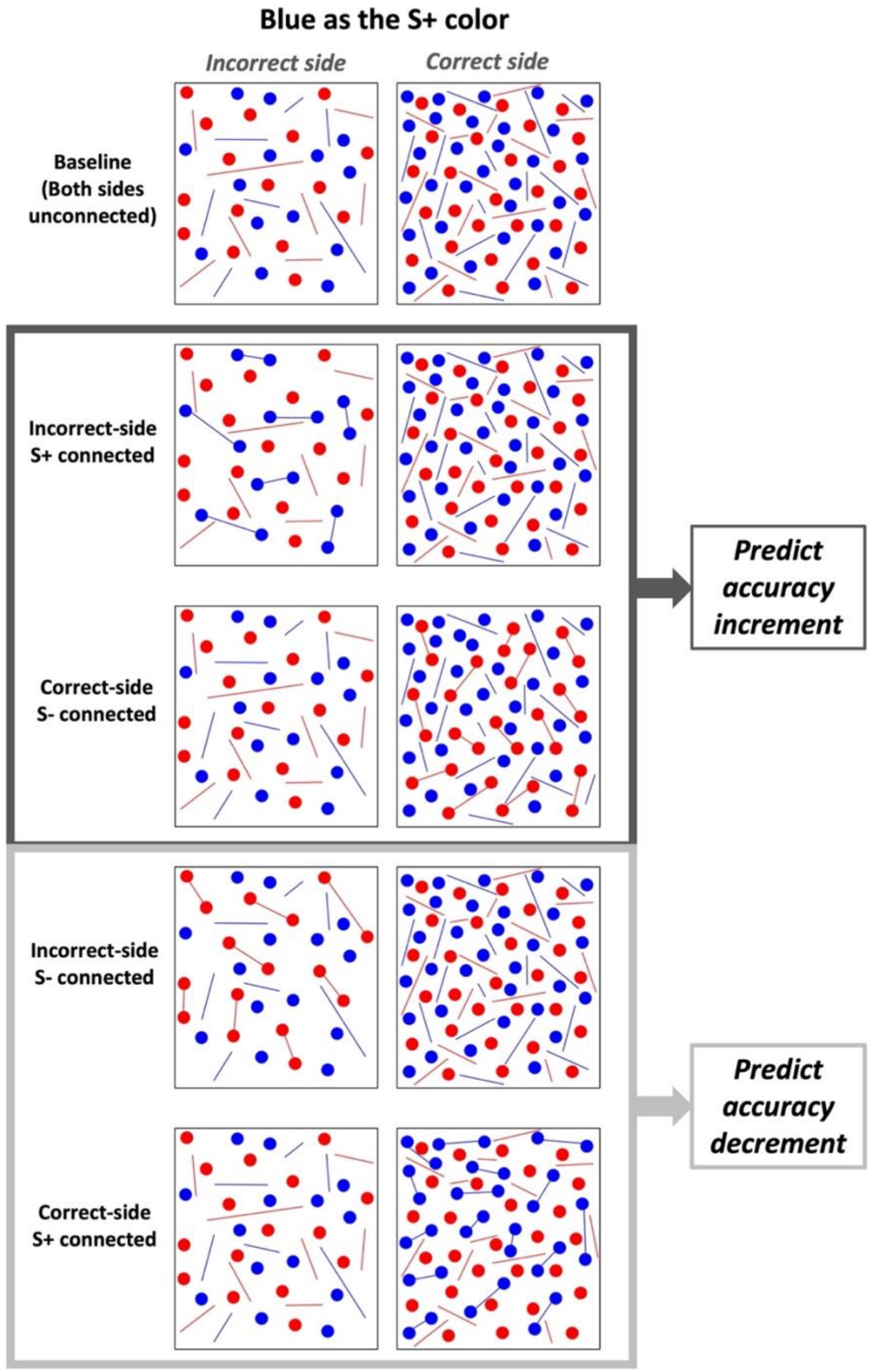

Fig. 1. Examples of standard stimuli for different connectedness conditions. When blue was assigned as the favorable color (S+), the side that contained a more favorable ratio of blue dots was referred to as the correct side, whereas the side that contained a less favorable ratio of blue dots was referred to as the incorrect side. There are three within-subject conditions: 1) Baseline condition where neither stimulus contained connected dots; 2) Predict-increment condition where the favorable dots (S+) on the incorrect side were connected or the undesirable dots (S-) on the correct side were connected; 3) Predict-decrement condition where the undesirable dots (S-) on the incorrect side were connected or the favorable dots (S+) on the correct side were connected.

Design and Procedure

Participants completed consent and demographic forms before the start of the experimental tasks. On each trial, participants first saw a central fixation point for 500-ms. Then two stimuli were presented simultaneously on the screen, one to the left of the fixation point and one to the right. Stimuli remained on the screen for 1500ms, after which the stimuli and fixation point disappeared, and participants were instructed to press 'f' if the left-hand array was deemed to have a higher probability of randomly producing a favorable item, and 'j' if the right-hand array was deemed to have a higher probability of randomly producing a favorable item.

Half of the participants were randomly assigned blue as the favorable target, with the other half assigned red as the favorable target. In all cases, participants were explicitly told to ignore the lines, to attend only to the number of dots, and to indicate "which box has a higher ratio of [favorable dots] to [unfavorable dots]". Prior to completing the task, participants completed an instruction block with multiple example trials, where the instruction to ignore lines was emphasized repeatedly and it was explained that whether lines happened to connect dots was irrelevant to the number of dots and, thus, irrelevant to the ratio of blue to red dots.

Results

Overall participants performed significantly above chance (Accuracy M = 76.63%, SD = 0.01, t(25) = 53.01, p < .000, 95% CI [0.74, 0.80]; one-sample t-test). To test whether the ratio of ratios among the two stimuli affected accuracy on baseline trials, we ran a generalized linear mixed-effects model (GLMM). This model followed a binomial distribution predicting participants' item-level accuracy with the ratio of ratios as a fixed effect and a random effect of individuals. As shown in Fig 2b, we found a significant fixed effect of the ratio of ratios (β_{RR}= 0.52, SE = 0.07, Z = 7.08, p < .000), indicating that accuracy parametrically varied based on the ratio of ratios between the two arrays, replicating Szkudlarek and Brannon's (2021) findings in elementary school children.

Our main pre-registered prediction was that accuracy would be lower on two of the trial types and higher on the other two trial types relative to baseline depending on whether the favorable (S+) dots were connected or the unfavorable (S-) dots were connected on the correct or incorrect side. We thus segregated the data into three within-subject conditions: 1) a Baseline Condition where there were no connections between dots; 2) a Predict Increment Condition where the favorable dots (S+) on the incorrect side were connected or the unfavorable dots (S-) on the correct side were connected; 3) a Predict Decrement Condition where the unfavorable dots (S-) on the incorrect side were connected or the favorable dots (S+) on the correct side were connected (See Figure 1). As shown in Fig. 2b, an ANOVA revealed a main effect of condition ($F(1.4,34.5) = 38.33$, $p < .000$, $\eta^2_p = .251$). We conducted post-hoc pairwise comparisons between the Baseline Condition and each of the other two conditions, averaging across other factors (e.g., the ratio of ratios of the two stimuli, the assigned favorable colors). As predicted, when compared to the Baseline Condition which had no connecting lines (Accuracy M = 78.3%), the accuracy of ratio comparisons increased in the Predict Increment condition (Accuracy M = 81.9%, $t(25) = 4.04$, $p = .001$, Cohen's $d = 0.79$). Conversely, when compared to the Baseline Condition, participants were significantly less accurate in the Predict Decrement condition (Accuracy M = 70.6%, $t(25) = 6.06$, $p < .000$, Cohen's $d = 1.19$).

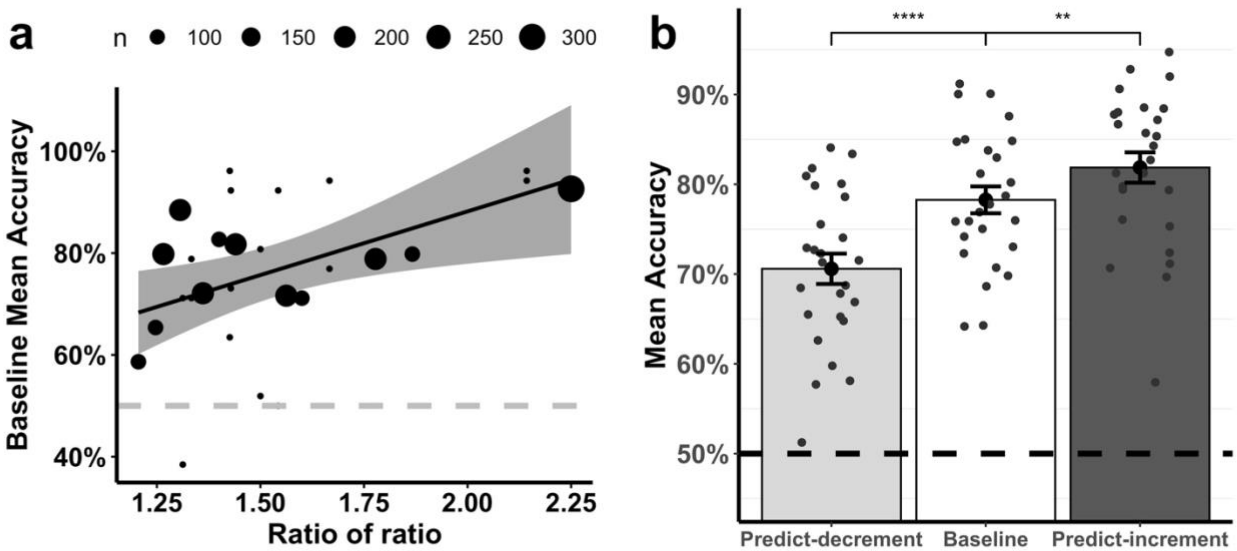

Fig. 2. Results of Experiment 1. (a) Scatter plot of mean accuracy in the baseline condition as a function of the ratio of ratios of the two stimuli. Performance on the ratio comparison task was dependent on the ratio of ratios. The grey dashed line indicates chance accuracy. The size of points denotes the number of trials in a visually proportional way. **(b)** Bar plot of mean accuracy on ratio comparisons in the different connectedness conditions. All error bars are *SEM*. Dashed line denotes chance accuracy. **p < .01, ***p < .001, ****p < .0001.

Discussion

Experiment 1 replicates prior work showing that humans can perform fast and intuitive visual ratio comparisons. In line with previous results from young children (Szkudlarek & Brannon 2021), these ratio comparisons were imprecise, or approximate, with accuracy predicted by Weber's Law (i.e., the ratio among the ratios compared). In this respect, performance in our task mirrored the distinctive performance profile of the natural number discriminations facilitated by the ANS.

Experiment 1 extended this existing work by showing that our ability to accurately discriminate ratios is susceptible to the numerical "connectedness illusion" (He et al. 2009; Franconerri et al. 2009). When favorable items on the correct side (defined based on the true number of dots red and blue dots regardless of connections), or non-favorable items on the incorrect side, were connected with thin lines, participants performed worse than in an otherwise identical baseline condition where all elements were left unconnected. Conversely, when non-favorable items on the correct side, or favorable items on the incorrect side were connected with thin lines, participants performed better than in an otherwise identical baseline condition where all elements were left unconnected. In both cases, this is what we would expect if connections among items leads to a reduction in their perceived number, with this (in turn) influencing the representation of the ratio/fraction among colored items.

Such results are congenial to Clarke and Beck's (2021a) hypothesis that ratios or rational numbers are extracted via the ANS's prior extraction of natural numbers (e.g., the natural number of items in the subsets/supersets whose ratio is then compared). It is well established that the ANS's extraction of natural numbers is affected by the connectedness illusion, such that connecting two dots with a thin line (effectively turning two dots into a single dumbbell-shaped object) leads to a reduction in their perceived number. As such, Clarke and Beck's hypothesis predicts that connections among dots would affect perceived ratios/fractions in the above respects. Meanwhile, alternative architectures which deny that rational number extraction is based on the ANS's prior extraction of natural numbers, and instead posit multiple routes to number extraction (e.g., Matthews et al. 2016; Hubbard & Matthews 2021; Park, Viegut and Matthews 2020), make no such prediction.

Somewhat independently, our results provide evidence that the ratios/rational numbers on which comparisons are based pertain to the ratios/fractions among *numbers* of items in observed arrays, rather than those items' continuous (or non-numerical) properties. For a start, connectedness has a negligible effect on the non-numerical quantities that are observed in a seen collection. In baseline conditions from our study, dots were unconnected and lines were free-floating. Lines were, however, still populating the array. As a result, having thin lines connect dots made little difference to the total surface area of the items contained in the collections (i.e., the total number of blue and/or red pixels on the screen), or the total brightness or density of the displays. Consequently, it is hard to see how our results could be explained by mechanisms that simply track and represent ratios/fractions among continuous dimensions. By contrast, connecting two dots with a thin line *does* significantly affect the number of bounded items in a display, since it effectively turns two items into a single dumbbell-shaped object. Since our results show that connecting preferred or non-preferred items in a more/less favorable ratio affects the perceived favorability of these ratios in precisely the ways one would predict if connectedness were causing a reduction in the perceived number of items in relevant subsets, our results demonstrate that approximate ratios/fractions are extracted via mechanisms which function to extract the natural number of bounded objects. As such, our results suggest that subsequent ratio/fraction representations are robustly numerical, concerning ratios/fractions among natural numbers of items.

We find converging evidence for this numerical interpretation when we reflect on the fact that numbers are distinctive for having a *second-order* character; i.e., in that they can only be assigned relative to a sortal (Frege 1910). To illustrate, suppose that I gesture at a collection of boots and ask "How many?". Philosophers would note that this question is ill-posed. To answer it, we need to decide whether we're interested in enumerating the individual boots, the discrete pairs of boots, or the different boot types contained in the collection. For instance, 16 individual boots might only constitute 8 pairs of boots, or 1 type of boot. Hence, the way in which we individuate the items has an impact on the number or numerical quantity we assign to the collection. By contrast, if we want to know how much the boots weigh, or what their total volume is, it won't make any difference how the collection is carved up: irrespective of whether the collection is thought of as constituting a bunch of individual boots, pairs of boots, or types of boots, it will take up the same amount of space in my closet and register the same weight on a set of scales. (Of course, we could add or ignore a single boot, with this affecting the collection's weight/volume, but this involves us changing the collection itself – not simply changing how items within the collection are individuated.)

This is significant, since the effects of connectedness found in our experiment indicate that the extraction of ratios/fractions depends on the prior extraction of quantities whose values have a second-order character of this sort. This is because, *prima facie,* the reduction in perceived quantity associated with the connectedness illusion reflects the fact that the ANS functions to (somewhat inflexibly) enumerate whole bounded objects, even when this is detrimental to task performance. Thus, the system seems to be taking a stand on how the items it enumerates are to be individuated, with this influencing the value(s) it assigns to the collection. In this way, the effects of connectedness found in our study indicate that ratio/fraction comparisons are based on the outputs of an initial stage of processing which functions to extract quantities with a second-order character, distinctive of number or numerical quantity. This is, again, in line with Clarke and Beck's (2021a) suggestion that rational numbers are extracted by the approximate number system, via its prior extraction of natural number.

Experiment 2

Experiment 1 replicated previous work showing that humans can perform fast and intuitive ratio discriminations, albeit imprecisely and in accord with Weber's Law. Moreover, it extends this previous work by showing that ratio discriminations concern ratios of preferred to non-preferred *numbers* of items in displays, and are susceptible to distinctive signature limits of approximate number representation. But while these findings are, thus, consistent with the suggestion that rational numbers are extracted by a secondary stage of ANS processing which extracts and represents ratios/fractions based on that system's prior extraction of natural number, it may be questioned whether ratio comparisons of this sort are driven by the ANS itself (*pace* Clarke & Beck 2021a). For instance, it could be that ratio comparisons result from participants' domain general reflection on the relationship between the natural numbers of items that their ANS has extracted – i.e., the number of blue items, red items, or red and blue items in each ratio – not least, because the ANS's characteristic imprecision in representing natural numbers could potentially explain why subsequent ratio comparisons then conform to Weber's Law in the way we observed (Dramkin & Odic 2021). To address this concern, Experiment 2 sought to replicate Experiment 1 while requiring participants to engage in an additional verbal shadowing task, which was designed to consume verbal working memory and prevent explicit counting/arithmetic. We reasoned that, if ratios are being represented and processed by domain general cognition, a distractor task that taxes verbal working memory should impair performance. Conversely, if ratio comparisons are performed by independent processes, such as a relatively autonomous ANS (and hence independently of flexible thought or verbal reasoning), we should expect the addition of this secondary task to have little effect on performance.

Method

Participants:

31 undergraduate students (Mean age = 19.78 years, SD = 1.18 years, 18 females) participated in this experiment in exchange for course credit and were tested in a university laboratory. Participants provided informed consent to a protocol approved by the Institutional Review Board of a large research university before starting the experiment. All participants reported normal or corrected-to-normal visual acuity and color vision (as in Experiment 1, color sightedness was also independently verified in an online test). Following the pre-registered exclusion criteria, 1 participant was removed due to incomplete data. 1 participant was excluded because they exhibited a side bias where the right-side arrays were chosen on greater than 65% trials. 5 participants were removed because they did not perform above chance expectations on baseline trials. 1 participant identified as an extreme outlier was removed. The final sample size was 24 (Mean age = 19.79, SD = 1.22; 14 females).

Materials and design:

Experiment 2 was designed to be as similar as possible to Experiment 1 except for the following deviations. Unlike Experiment 1, participants were required to perform an additional verbal shadowing task throughout the entire experiment. This involved participants repeating the words “Mary had a little lamb, her fleece was white as snow” over and over, whilst performing the ratio discrimination task. To ensure that this additional verbal shadowing task was performed, participants were tested in a laboratory testing room with an experimenter monitoring their verbal recital, rather than being tested online. Before starting the experiment, participants practiced repeating the assigned sentence until they felt comfortable performing the verbal shadowing task. An experimenter monitored each participant’s recitation throughout the experiment and was prepared to tap their shoulder if they paused the recitation. This only happened on one occasion and was quickly rectified. The same experimental program used in Experiment 1 was run on a Dell personal computer (Peirce, 2007). The monitor was 51 × 28.5 cm with a resolution of 1,920 ×1,080 pixels. Participants were at an average viewing distance of 56 cm from the screen. Otherwise, the design and stimuli used in the experiment were identical to that of Experiment 1.

Results

As in Experiment 1, accuracy was significantly above chance despite the requirement to perform an additional verbal shadowing task throughout (Accuracy M = 75.34%, SD = 0.01, t(23) = 58.38, p < .000, 95% CI [0.73, 0.78]; one-sample t-test). As in Experiment 1, we found a significant effect of connectedness on task accuracy (F(2,46) = 35.32, p < .000, η^2_p = .249). As shown in Fig 3b, accuracy was higher in the Predict Increment Condition compared to the Baseline Condition (t(23) = 4.63, p < .001, Cohen's d = 0.95) and lower in the Predict Decrement Condition compared to the Baseline Condition (t(23) = 4.81, p < .001, Cohen's d = 0.98). As shown in Fig 3a and consistent with Experiment 1, the ratio of ratios of the two arrays parametrically affected the accuracy of ratio comparisons. A GLMM controlling for the random effect of individuals revealed a significant fixed effect of the ratio of ratios on participants' item-level accuracy in the baseline condition (β_{RR}= 0.86, SE = 0.10, Z = 8.86, p < .000), indicating that the approximate ratio discriminations conformed to Weber's Law.

To test whether the addition of our verbal shadowing task impaired ratio discriminations, we analyzed the data from participants tested in the dual-task paradigm used in Experiment 2 together with the data from participants tested in the single task paradigm from Experiment 1. We conducted a GLMM adding task format (with and without verbal shadowing) entered as a fixed effect while controlling for the fixed effect of the three connectedness conditions as well as the random effect of individuals. This GLMM followed a binomial error distribution with a logit link function. Consistent with our pre-registered prediction, we found no significant fixed effect of task format on participants' accuracy as a function of condition (β_{task}= -0.13, SE = 0.11, Z = -1.12, p = .26). This indicates that the addition of a secondary verbal shadowing task did not significantly disrupt approximate ratio comparisons.

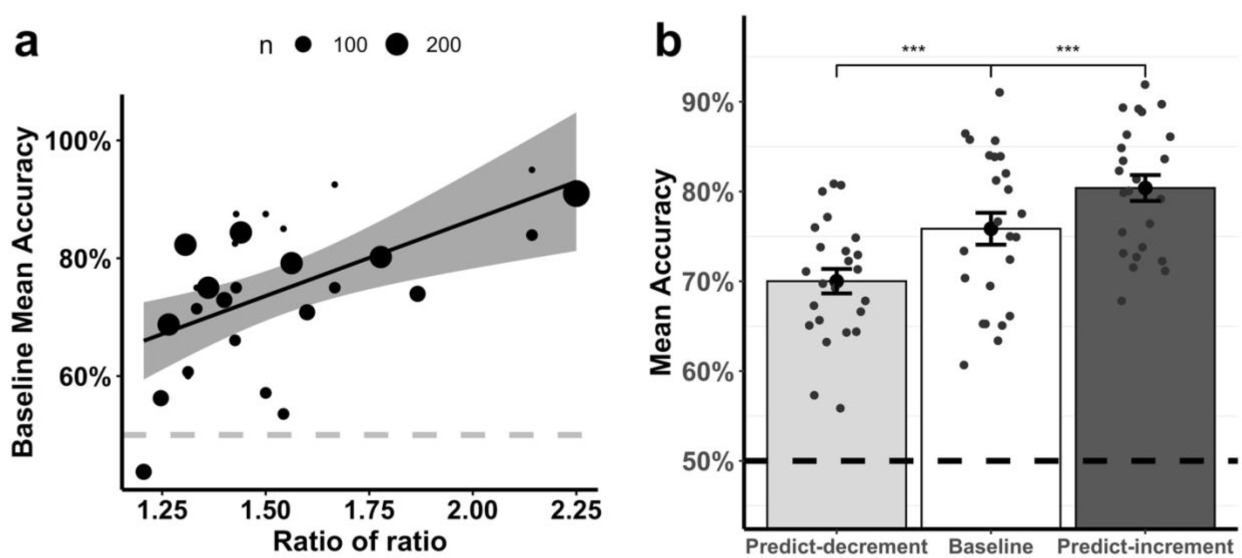

Fig. 3. Results of Experiment 2. Adding a verbal shadowing task did not significantly affect participants' performance on ratio discriminations compared to Experiment 1. The conventions of plots in (a) and (b) are the same as in Fig. 2.

Since the addition of a secondary verbal shadowing task did not significantly impact performance in Experiment 2 compared to Experiment 1, we then combined data from Experiment 2 with data from Experiment 1 (n=50) to increase power and conduct more detailed exploratory analysis of the factors contributing to task performance.

Heuristic analyses

Previous work with our experimental task and similar tasks suggests that performance is largely driven by the ratio of ratios, but that choices are biased by three heuristics that depend on the absolute number of items in the observed collections (Szkudlarek & Brannon 2021; Falk et al 2012; O'Grady & Xu 2020). The "More Good" strategy involves participants choosing the ratio with the greatest number of favorable items. The "Less Bad" strategy involves participants choosing the ratio with the smallest number of unfavorable items. Finally, the "More Items" strategy involves participants choosing the ratio with the greater total number of items.

Following Szkudlarek and Brannon (2021) we assessed the contribution of these heuristics to participants' performance in our experiments. We first created a Ratio Model in which we identified for each trial whether using a true ratio strategy would lead to a left-side response (coded as 0) or a right-side response (coded as 1). We then constructed a model for each of the incorrect heuristic strategies. Each heuristic model indicated the left or right responses people would make if they used that heuristic (1 for choosing the right side and 0 for choosing the left side). To identify whether participants' responses deviated from each heuristic, we subtracted each heuristic model from participants' actual left and right choices and took the absolute value. This was referred to as the Actual Deviation from Heuristic Model (0 for responses aligned with each heuristic, 1 for responses different from that predicted by a given heuristic). Next, we identified whether responses predicted by the true Ratio Model deviated from that predicted by each Heuristic Model for which we created the Ratio Deviation from Heuristic Model. This was calculated as the absolute difference values between each Heuristic Model and the Ratio Model. For a single trial, a 0 indicated that the Ratio Model and a given Heuristic Model predicted the same response whereas a 1 indicated that the Ratio Model predicted an opposite response from a given Heuristic Model.

To examine whether participants performed true ratio discriminations independently of the abovementioned heuristics, we ran a GLMM separately for each heuristic following a binomial distribution predicting participants' Actual Deviation from Heuristic Model as a function of the fixed effect of Ratio Deviation from Heuristic Model and the random effect of participants. If a participant was exclusively relying on a given heuristic to perform the task, then the Actual Deviation from Heuristic Model would mostly indicate 0s and so the β value of the fixed effect should be close to zero. In contrast, a significantly positive β value would indicate that the Ratio Deviation from Heuristic Model better explained participants' performance. The three GLMMs for each of the three heuristics all revealed significant fixed effects of the Ratio

Deviation from Heuristic Model, indicating choices predicted by the true Ratio Model when controlling for the use of each heuristic ("More Good": β_{RD}= 2.38, SE = 0.07, Z = 35.05, p < .000; "Less Bad": β_{RD}= 2.69, SE = 0.07, Z = 36.44, p < .000; "More Item": β_{RD}= 2.71, SE = 0.07, Z = 37.93, p < .000). We thus found support for the ratio strategy controlling for each heuristic.

Although the above analysis demonstrates that adults did not exclusively rely on any of the heuristic strategies to perform ratio discriminations, it remains possible that the errors adults made were predicted by one or more heuristic. To test this, we constructed a GLMM for each heuristic predicting participants' actual errors on each trial with the Ratio Deviation from Heuristic Model entered as the fixed effect while controlling for the random effect of participants. A positive β value would indicate that people's errors were systematically predicted by the use of a given heuristic. We obtained significantly positive β values for the "More Good" strategy (β_{RD}= 0.87, SE = 0.07, Z = 12.80, p < .000) and the "More Item" heuristic (β_{RD}= 0.83, SE = 0.07, Z = 12.38, p < .000). We obtained a negative β value for the "Less Bad" heuristic (β_{RD}= -0.64, SE = 0.07, Z = -9.11, p < .000) indicating that people's actual errors were the opposite of the responses predicted by choosing the array with fewer unfavorable items. In sum, our strategy analysis replicates prior work in children, providing evidence that participants use a true ratio comparison strategy when successfully discriminating seen ratios, and that errors in these tasks are more often explained by the "More Good" and "More Item" heuristics (Szkudlarek & Brannon, 2021).

The relative salience of favorable vs unfavorable items

Next, we performed a more detailed analysis of how connectedness influenced ratio processing, again using the combined data set from Exp. 1 and Exp. 2. Figure 4 displays all 5 trial types without collapsing the trial-types into three conditions. As predicted, accuracy increased for both predict increment trial types ('Correct-U' and 'Incorrect-F') and decreased for both predict decrement trial types ('Incorrect-U' and 'Correct-F') relative to baseline.

When favorable items on the incorrect side were connected, accuracy increased to 83.1% compared to 76% on baseline trials (t(49) = 5.91, p < .000, Cohen's d = 0.84). Meanwhile, when unfavorable items on the correct side were connected accuracy increased to 79.2% (t(49) = 2.80, p < .01, Cohen's d = 0.40). Conversely, accuracy decreased to 73.5% when unfavorable items on the incorrect side were connected again compared to 76% on baseline trials (t(49) = 3.53, p < .01, Cohen's d = 0.50). Meanwhile, when favorable items on the correct side were connected accuracy decreased to 67.1% (t(49) = 8.54, p < .000, Cohen's d = 1.21). Thus, our main prediction that connectedness would systematically affect perceived number and, in turn, ratio comparison was supported even when analyzing at the level of trial-type.

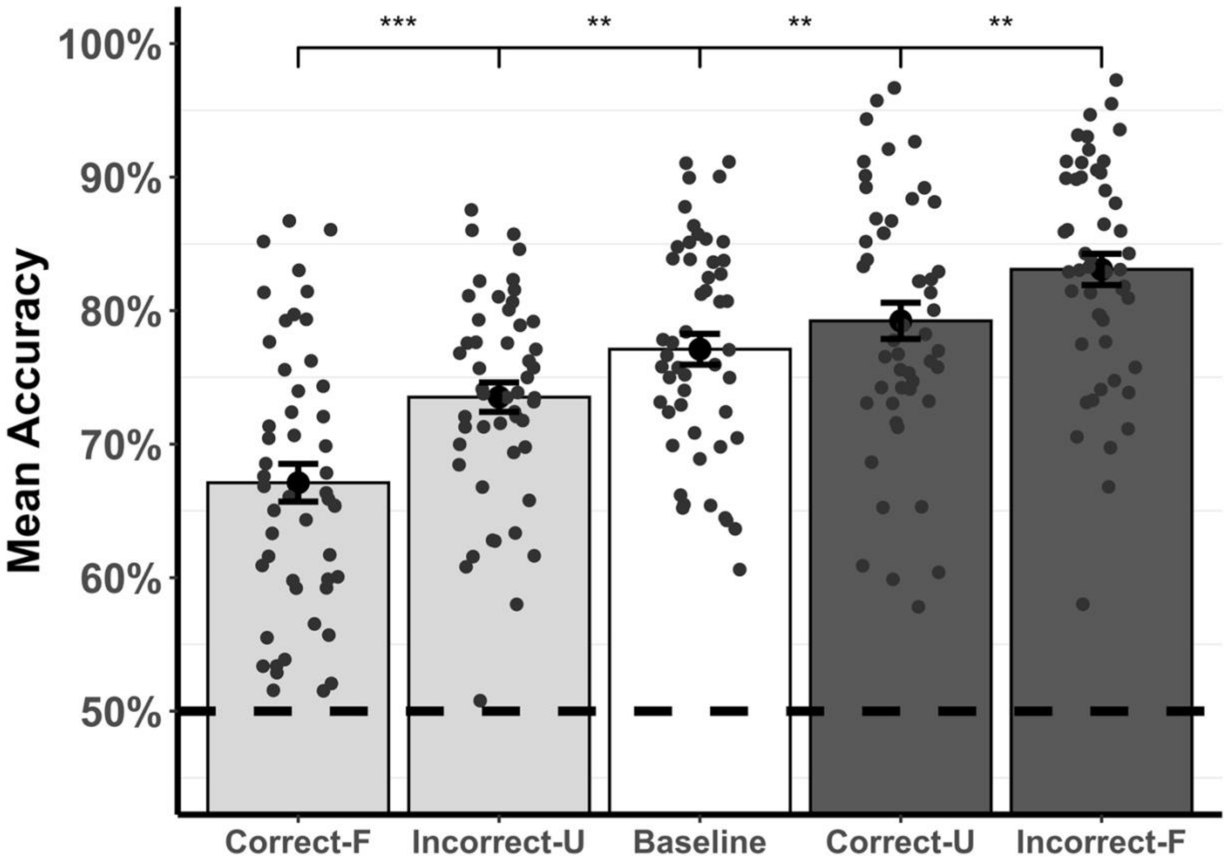

Fig. 4. Bar plot of mean accuracy on ratio comparisons across different connectedness trial types. All error bars are *SEM*. Dashed line denotes chance accuracy. Asterisks indicate the level of significance between two adjacent bars. **p < .01, ***p < .001.

Interestingly, however, connecting favorable items, had a larger impact on perceived ratio than connecting unfavorable items. Although both Predict-Increment conditions were greater than baseline, accuracy was higher when favorable items on the incorrect side were connected (83.1%) compared to when unfavorable items on the correct side (79.2%) were connected ($t(49) = 3.18$, $p < .01$, Cohen's $d = 0.45$). Similarly, although both trial types within the Predict-Decrement Condition were lower than baseline, accuracy when favorable items on the correct side (67.1%) were connected was significantly lower compared to when unfavorable items on the incorrect side (73.5%) were connected ($t(49) = 4.89$, $p < .000$, Cohen's $d = 0.69$).

Discussion

Experiment 2 replicated the results of Experiment 1 while requiring that participants simultaneously perform an additional verbal shadowing task, designed to consume verbal working memory, and prevent the use of explicit counting routines, mental arithmetic, or analogical reasoning (e.g., Waltz et al. 2000). Under these conditions, participants' performance continued to be predicted by the ratio among the ratios of items, again highlighting that the accuracy of ratio comparisons is predicted by Weber's Law. As in

Experiment 1, we found that connecting target vs non-target items in a ratio systematically altered their perceived number and, thus, ratio discriminations accordingly. Finally, and most importantly, performance in Experiment 2 was not significantly impaired relative to performance in Experiment 1, despite the addition of the secondary verbal shadow task.

These results are consistent with Clarke and Beck's (2021a) conjecture that ratio comparisons are facilitated by the ANS. If ratios/fraction representations are simply constructed by domain general resources which depend on verbal working memory we would expect to find a reduction in task performance in Experiment 2 as compared with Experiment 1, due to the addition of the secondary verbal shadowing task. This is not what we found. Rather, our results are what we would expect if numerical ratios/fractions were extracted by mechanisms with the computational profile of an ANS, operating somewhat independently of flexible thought/central cognition and in a relatively encapsulated manner.

Such an interpretation would be undermined if it turned out that performance was driven by heuristics, which did not involve any appreciation for ratios/fractions whatsoever. For instance, if it turned out that participants were simply selecting the more favorable of two ratios by selecting arrays with the larger absolute number of favorable items (the More Good strategy), the smaller absolute number of unfavorable items (the Less Bad strategy), or the larger absolute number of favorable and unfavorable items (the More Items strategy). However, our post-hoc analyses (collapsing over the results of Experiments 1 and 2) revealed that the ratio discriminations participants performed could not be fully explained by any one of these heuristics. Rather, performance was best explained by the ratio of ratios among favorable and unfavorable items (though errors were often explained by participants' adoption of erroneous More Good or More Item strategies). In each of these respects, our results replicate effects we observed in previous work with children (Szkudlarek & Brannon, 2021).

Our exploratory analyses may also provide a window onto the underlying format of the representations that drive task performance. To illustrate, notice that, in principle, performance in Experiments 1 and 2 could be explained in at least two ways. Firstly, task performance could be underwritten by *representations of ratio.* On this view, the underlying representations would specify the relationship between the blue and red subsets of the displays (e.g., the ratio 1:3 specifies that for every 1 blue there are 3 reds in the collection). Alternatively, task performance might be underwritten by representations specifying the relationship between the value of some privileged subset of the collection (e.g., the favorable or, perhaps, non-favorable items in each collection) and the value of the superset (i.e., the total number of blue *and* red items in the collection and, hence, the collection as a whole). On this second view, the representations would have the form of *a fraction* (e.g., ¼), wherein one subset is prioritized, and a denominator specifies the value of the superset. Simply put: people might be relying on part-part (e.g., ratio) or part-whole (e.g., fraction) representations in the ratio comparison tasks under consideration.

Prior work has suggested that children are initially biased to approach probability tasks with part:part representations (Smith 1986; Singer & Resnick 1992; Spinillo & Bryant 1999). In fact, children often have difficulty comparing proportions or probabilities because they focus on the parts rather than the whole and interventions that aide children in emphasizing the whole improve performance (Jeong et al. 2007). However, virtually all this prior work probes children's verbal reasoning about proportionality. It, thus, remains unclear whether a part:part strategy is the default strategy for nonsymbolic tasks. Non-verbal representations of proportionality need not mirror the strategies and representational formats used when children engage in *verbal* reasoning about proportionality.

Our exploratory analyses revealed that connecting the favorable items in either the correct or incorrect ratio had a larger effect on adults' performance than connecting the unfavorable items. One explanation for this result is that participants were employing proportional representations couched in a part-whole representational format (like a fraction or proportion) rather than a part-part format (like a genuine ratio). To illustrate, consider an array containing 10 blue circles and 10 red circles where blue is the favorable color. Suppose that the proportion of blues is represented using a part-whole format (like the fraction, 10 blues/20 blues and reds). In this scenario, connecting blue items would reduce their perceived number, affecting the fraction of whole bounded blue items in the collection accordingly. Boldly stated, it would reduce the number of bounded blue items (referred to by the fraction's numerator) to 5, and the actual number of bounded blue and red items (referred to by the fraction's denominator) to 15. Thus, it would change the actual fraction of bounded blue items from 1/2 (10blues/20 blue and reds) to 1/3 (5blues/15 blue and reds). Meanwhile, connecting the red items would leave the number of bounded blue items referred to by the fraction's numerator unaffected (there would still be 10 bounded blue items in the collection). It would, however, lead to a reduction in the number referred to by the fraction's denominator, specifying the value of the superset (reducing this from 20 to 15). Hence, connections among red items would change the actual fraction of bounded blue items in the collection from 1/2 (10blues/20 blue and reds) to 10/15, or 2/3. We can summarize these observations by noting that connections among blue items would change the actual fraction of bounded blue items in the array from 10/20 (or ½) to 5/15 (or 1/3), while connecting the red items will change the actual fraction of bounded blue items from 10/20 (or ½) to 10/15 (or 2/3).

While 1/3 and 2/3 both differ from ½ by the same *absolute* amount (1/6), the ratio among these ratios differs markedly: where 1/3 differs from ½ by a ratio of 1.5, 1/2 differs from 2/3 by a ratio of just 1.3. Given that the ratio among two ratios predicts discriminability, the conjecture that arrays are represented in a part-whole format, specifying the proportion of favourable items over the value of the superset, predicts that connections among favorable items will exert a greater effect on the perceived favourability of a ratio. As noted, this is precisely what our exploratory analyses revealed, and accords with prior work showing that when adults, infants and children engage in approximate enumeration tasks they automatically encode the quantity of the superset under varied attentional conditions (Halberda, Sires, & Feigenson 2006; Poltoratski & Xu 2013).

By contrast, suppose that the proportion of blue circles were represented as a ratio, with a part-part (or subset:subset) structure that specifies the number of blues to the number of reds. Here, connecting blue items would alter the ratio of bounded blue items to bounded red items from a value of 10:10 (or 1:1) to 5:10 (or 1:2); meanwhile connecting the reds would alter the ratio from 10:10 (or 1:1) to 10:5 (or 2:1). In this case the ratio of ratios will be equally different when favorable and unfavorable items are connected.

Such considerations reveal that the asymmetry found in our exploratory analyses can be straightforwardly explained by the hypothesis that task performance is underwritten by proportional representations with a part-whole format (like a fraction) as opposed to representations with a part-part structure (like a ratio).

There are however alternative explanations for the observed asymmetry is our results. The observed asymmetry found in our exploratory analyses may instead result from the effects of attention. Visual attention is known to play an important role in visual feature integration (Treisman & Gelade 1980; Wolfe 2012). Thus it is possible that participants attended more to the favourable items in observed displays, and this alone caused the favourable items to be bound into unified dumbbells more often than items in the un-favourable subsets. Indeed, this alternative suggestion is consistent with previous work indicating that the deployment of visual attention can affect the strength of the connectedness illusion (Pomé et al. 2021).

Future work is necessary to uncover the format of the representations that underlie our task performance. Part:part and part:whole representations are equally useful in the current task and the asymmetry we observed where connecting favorable items impacted performance more than unfavorable items is not sufficient to differentiate these structures. Moreover if the asymmetry does reflect a part:whole structure an open question remains as to whether this changes over development.

General Discussion

Across two experiments we examined adult humans' capacity to extract, represent, and compare the ratios/fractions of items in two displays. In line with previous findings, the accuracy of participants' discriminations was predicted by Weber's Law, such that the ratio among the ratios under comparison predicted task performance. Our results have extended this observation by showing that these ratio discriminations are (a) susceptible to a connectedness illusion, with connectedness reducing the perceived quantity of items in relevant subsets/supersets (with this then influencing ratio discriminations accordingly) and that they are (b) not significantly affected by the simultaneous performance of a secondary verbal shadowing task. Subsequent analyses showed that (c) performance in these tasks could not be explained by simpler heuristic strategies that do not involve any sensitivity to the ratios/fractions each display portrayed, with (d) the effects of connectedness among target items exerting a greater change in performance than connections among non-target items.

How do our findings bear on the hypothesis that the ANS represents rational numbers (Clarke and Beck 2021a)? Our main finding was that connectedness systematically impacted task performance. This finding indicates that number is the relevant dimension driving task performance, and suggests that the ratios/rational numbers that are being represented concern ratios/fractions among discrete quantities, as extracted by the ANS. Thus, approximate number representations of the natural number of red and blue items are formed (or, alternatively, representations of the natural number of favorable items and of items in the superset are formed) and rational number representations are subsequently formed and compared on the basis of these.

An alternative to this proposal would be that ratio discriminations are achieved by processes which bypass the ANS, and its representations of natural number, entirely (Hubbard & Matthews 2021). A challenge for such an alternative is, however, that of providing some principled explanation for the effects of connectedness observed in our study. For while there may be multiple routes to visual number extraction, current evidence indicates that these are not all susceptible to the connectedness illusion. For instance, Fornaciai and Park (2021) found that representations of number in the early visual cortex are not affected by connectedness in the way that approximate number representations implemented in the intraparietal sulcus are known to be. The conjecture that rational numbers are extracted via ANS representations of natural number thus offers a simple and parsimonious explanation for the effects of connectedness documented in our study. It is also supported by recent work in machine learning, where deep convolutional neural networks that have been trained to compare numerical quantities in a range of distinct formats seem to base their assessment of rational numbers on "ANS style" units responding to natural number (Chuang et al. 2020).

Several questions remain open. Finding that the completion of a secondary verbal distractor task failed to impact task performance (Experiment 2) implies that these ratio discriminations do not depend on verbal comparisons or explicit counting routines. As noted, this is consistent with the view that rational numbers are extracted by the ANS itself, based on its prior extraction of natural number (Clarke & Beck 2021a). However, it is also consistent with the view that ratios or rational numbers are being extracted by a domain-general *ratio processing system* (RPS), that compares ratios/fractions among both numerical and non-numerical quantities (Matthews et al. 2015); at least insofar as the RPS avoids placing a burden on verbal working memory in a manner that renders it akin to the ANS proper (see: Cordes et al. 2001).

Support for the RPS proposal comes from studies demonstrating that children and adults can compare ratios across distinct magnitude types, such as line length and number (Binzak, Matthews, & Hubbard 2019). For instance, Bonn and Cantlon (2017) found that subjects spontaneously compare ratios across magnitude types when judging sequences as more or less similar. On their account, this indicates that ratios are represented in "a highly general code that could qualify as a candidate for a generalized magnitude representation". Meanwhile, Park, Viegut and Matthews (2020) found that ratio acuity for stimuli in one format (e.g., line ratios) was a better predictor of ratio acuity in other stimulus formats (e.g., numerical ratios), than performance in a non-ratio task with the same stimulus format (e.g., ratio acuity for line length and ordinal comparison for line length). Such findings have been taken to suggest that there are shared cognitive processes involved in the processing of ratios among line lengths and numbers.

On the other hand, these same researchers also found differences in the processing of discrete and continuous magnitudes. For instance, in both ordinal and ratio comparisons, numerical discriminations were found to be harder than size judgments with both circles and irregular shapes (ibid.). Prior work also demonstrates significant differences in the developmental trajectory of proportional reasoning about continuous and discrete quantities (Boyer, Levine, & Huttenlocher 2008; Jeong et al., 2007; Singer-Freeman & Goswami 2001; Spinillo & Bryant 1999) and suggests that children employ different strategies when *reasoning* about numerical and non-numerical ratios and fractions (Smith 1986; Hurst & Cordes 2018; Jeong et al. 2007). These latter results are, thus, suggestive of some degree of psychological independence in the extraction and representation of ratios/fractions among natural numbers and non-numerical dimensions. Given these mixed results, further work is needed to assess the degree of domain generality exhibited by the mechanisms of rational number extraction implicated in ratio comparison tasks.

One of the more intriguing aspects of the present study was the finding revealed by our exploratory analyses that connectedness among favorable items influenced ratio discriminations to a greater extent than connectedness among non-favorable items. As noted in the discussion to Experiment 2, this pattern of results should be expected if task performance is driven by rational number representations with a subset-superset structure that is isomorphic to the format of a proportion or fraction, rather than that of a ratio with a subset-subset structure. If this were the case it would be significant in light of critics of the view that the ANS represents rational numbers who have sometimes maintained that the system merely represents *ratios*, where ratios are to be distinguished from full-blown rational numbers (like fractions) along several dimensions (Ball 2021; Gomez 2021; Lyons 2021; Peacocke 2021). Nevertheless, the suggestion that the ANS might represent rational numbers, with a subset/superset format is somewhat surprising given the difficulties children have learning to manipulate and compare symbolic fractions (Lortie-Forgues, Tian, & Siegler, 2015). Furthermore, it could seem to be in tension with older work indicating that children consistently erroneously rely on subset-subset representations when engaging in proportional reasoning tasks (Smith 1986; Singer & Resnick 1992; Spinillo & Bryant 1999). For instance, Singer and Resnick (1992) tested 6th, 7th, and 8th graders on "fifteen probability problems which varied information content and quantitative relationships between the quantities expressed" (231) and found that children consistently indicated use of a part-part strategy when required to "think-aloud as they tried to solve each problem" (237).

Why then these discrepancies? For the purposes of answering these questions, it is a limitation of the present study that we tested adults not children. However, findings from Szkudlarek and Brannon (2021) already suggested that our task may allow children to avoid some of the biases they fall prone to in other tasks. While 6-8 year-old children did exhibit a tendency to use the "more good" heuristic when their performance deviated from the correct ratio strategy, they did not exclusively rely on such a strategy as they did in other contexts (Jeong et al. 2007). Moreover, many of the paradigms that revealed a strong "more good" strategy in children employed probabilistic reasoning tasks that relied on verbal strategies. For example, Singer and Resnick required children to explicitly "think-aloud". Verbal reasoning about probability appears to tap fundamentally different cognitive resources from the approximate visual ratio comparisons used in the present study, given that performance in our task was unaffected by verbal shadowing. Thus, even if the ANS represents rational numbers, allowing for intuitive comparisons of proportionality early in development, children nevertheless face a formidable challenge mapping such primitive and intuitive representations onto the conceptual or linguistic representations of fractions employed in math class (this is akin to the more familiar challenge they face mapping number words onto ANS representations – see: Carey & Barner 2021). But as we acknowledged earlier, the asymmetry adults displayed (whereby connecting favorable items had a great impact on performance than connecting unfavorable items) may instead result from greater attention to favorable items enhancing the connectedness effect for these items rather than revealing anything about representational structure. Further work is needed to definitively test whether part:part or part/whole representational formats are spontaneously employed in numerical (and non-numerical) ratio discrimination tasks, and whether this changes over development.

Conclusion

We have reported the results of two experiments designed to probe the nature and workings of approximate visual ratio comparisons. Using a well-known 'connectedness illusion' (He et al. 2009; Franconerri et al. 2009), we manipulated perceived number in a ratio comparison task independently of confounding variables. Our findings suggest that ratio comparisons rely on representations of the natural number of perceived items, and do not place demands on verbal working memory. In addition, we demonstrated that while simple heuristics can explain the errors that adults make in these tasks, they do not rely on any single heuristic to bypass the task instructions and are sensitive to the proportion of favourable items in each collection. Finally, our exploratory analyses should inspire additional research to uncover whether subset:subset or subset-superset representational structures underlie task performance.

Overall, our findings are consistent with the hypothesis that the ANS supports the representation of full-blown rational numbers (Clarke and Beck's, 2021a).

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